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BOUNDS AND APPROXIMATIONS FOR A GENERALIZED MEASURE OF PERFORMA--ETC(U)
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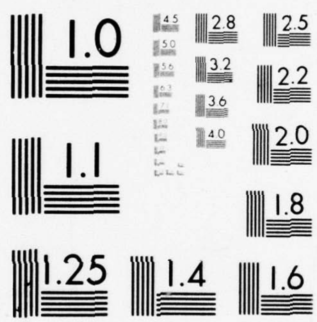

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BOUNDS AND APPROXIMATIONS TO A GENERALIZED MEASURE OF PERFORMANCE IN THE G/M/1/N QUEUE

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ABSTRACT

A generalized measure of performance is defined as a weighted combination of the ergodic queue length distribution where the weights are general functions of the system parameters. The paper presents a sequence of upper and lower bounds for this measure of performance in the G/M/1/N queue with FIFO discipline. The bounds are used to derive a sequence of approximations with bounded errors. The upper and lower bounds are shown to converge to their corresponding exact values. The technique used is based on the imbedded Markov chain analysis and considers only subsets of the steady state equations to derive the bounds. Initial computational experience is encouraging and has indicated that the approximations are viable for heavy and medium traffic conditions.

1. INTRODUCTION

Consider a single server queue with a finite capacity N and 'first come, first served' discipline. The arrival epochs form a recurrent process and an arriving customer leaves the system without service if the number of customers in the system is N. The service times are independent and identically distributed random variables with negative exponential distribution with mean μ^{-1}. In an extended Kendall notation a system of this type can be identified as G/M/1/N. This queue and its generalization G/M/s/N (with s servers) are combined loss and delay systems occuring in telephone trunking and communication problems.

1.1 Preliminary results:

Let $t_1, t_2, t_3, \ldots$ be the arrival epochs in the system, and $U_n = t_n - t_{n-1}$ be independent and identically distributed random variables with

$$P[U_n \leq x] = A(x),\ x \geq 0$$

and $E[U_n] = \lambda^{-1}$.

Let Q(t) and Q_n be the number of customers in the system at time t and just prior to the n^{th} arrival respectively. Because of the exponential nature of service times, $\{Q_n,\ n = 1,2,3,\ldots\}$ is a finite Markov chain imbedded in the stochastic process $\{Q(t),\ t \geq 0\}$ with transition probabilities P_{ij}, $0 \leq i, j \leq N$, where

$$P_{ij} = P[Q_{n+1} = j \mid Q_n = i]$$

$$= \int_0^\infty \frac{e^{-\mu x}\,(\mu x)^{i+1-j}}{(i+1-j)!}\,dA(x),\quad i+1 \geq j \geq 1,\ N > i \geq 0,$$

clearly,

$$P_{i0} = 1 - \sum_{j=1}^{i+1} P_{ij} \qquad N > i \geq 0$$

and $$P_{Nj} = P_{N-1,j} \qquad N \geq j \geq 0$$

Let $\underline{q}' = (q_0, q_1, \ldots, q_N)$ denote the steady state probabilities of the number of customers in the system just before an arrival occurs. Then it is well known that $\underline{q}$ is the unique solution to the system of equations

$$\underline{q}' \underline{P} = \underline{q}' , \ \underline{q}' \underline{1} = 1 \tag{1}$$

where $\underline{P}$ is the one step transition probability matrix and is given by:

$$\underline{P} = \begin{bmatrix} 1 - \alpha_0 & \alpha_0 & & & & \\ 1 - \alpha_0 - \alpha_1 & \alpha_1 & \alpha_0 & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ 1 - \sum_0^{N-1} \alpha_j & \alpha_{N-1} & \cdot & \cdot & \alpha_1 & \alpha_0 \\ 1 - \sum_0^{N-1} \alpha_j & \alpha_{N-1} & \cdot & \cdot & \alpha_1 & \alpha_0 \end{bmatrix}$$

where we have written

$$\alpha_j = \int_0^{\infty} e^{-\mu x} \frac{(\mu x)^j}{j!} \, dA(x) \tag{2}$$

The method of imbedded Markov chain as applied to G/M/1 is due to Kendall [11], and further use of the approach is presented in Prabhu [14], Keilson [10], Bhat [2], Laslett [13] and most recently Raju and Bhat [15].

1.2 A Generalized Measure of Performance

Measures of performance are functions of system parameters λ, μ, N, etc. They are needed in operational analysis where the best parameter values which optimize system performance are sought. Measures have been used since Erlang's work "The Rational Determination of the Number of Circuits" [3] where the probability of loss to the system is minimized. For the history and development of operational models and their measures see Bhat [1] and for a recent survey see Crabill et al [5]. Typical measures of performance are expressed in terms of cost of service per customer, waiting cost, lost customer cost, server utilization, etc. [Hillier [8]]. Since the most commonly used measures can be written as a linear combination of the steady state probabilities, a natural extension will be the generalized measure:

$$Z_N = \sum_{i=0}^{N} \bar{a}_i \, q_i \, , \qquad (3)$$

where $\{\bar{a}_i\}$ are general functions of the system parameters. For applications of this measure see Evans [6] and Kotiah [12].

This investigation considers the evaluation, bounding and approximation to the generalized measure as defined in (3) as a necessary step toward optimizing the system performance. Early experimentation with (3)

using some general distributions shows that Z_N is nonconvex, nonconcave, and even multimodal. Since optimizing a generalized function (as Z_N) is likely to call for a large number of functional evaluations, a fast technique to evaluate Z_N is needed. Moreover, the bounds can be used in a Branch-and-Bound framework, and the approximations can be utilized to obtain approximate optimal solutions. Existing analytical techniques are fairly complicated to be extended for operational analysis, but a technique as in [15] is promising for the fast numerical results that we seek.

The paper is divided into five sections. Section 2 investigates the ratio properties of the steady state distribution. Such properties will facilitate calculating the steady state distribution and reformulating the steady state equations to help develop the bounds. The suggested sequence of upper and lower bounds to Z_N is presented in section 3. A sequence of approximations that seems to work best for heavy and medium traffic is given in section 4. Computational experience for the approximation is given in section 5.

It should be clear that all the results presented here using the arrival epoch steady state distribution can be easily modified for the arbitrary time distribution. This is true by virtue of Hokstad's [9] result which shows that

$$q_K = \frac{\lambda}{\mu} \pi_{K-1} \text{ , } K = 1, 2, \ldots, N$$

where π_K is the arbitrary time steady state probability that there are K customers in the system [see also Takács [16] and Heyman [7]].

2. RATIO PROPERTIES OF THE STEADY STATE SOLUTION

In this section we focus on the properties of the ratio of steady state probabilities

$$r_i = q_{N-i+1}/q_{N-i} , \quad i = 1, 2, \ldots N.$$

We show that it suffices to generate a single sequence r_i, $i = 1, 2, \ldots, K$, independent of N, to calculate the steady state distribution for any of the queues

G/M/1/i, G/M/1/i + 1, ... , G/M/1/N, ... with the same general inter-arrival and exponential service time distributions.

<u>Theorem 1</u>: The arrival epoch steady state distribution $\{q_i\}_{i=0}^{N}$ of the G/M/1/N has the form

$$q_0 = c_0$$

$$q_i = c_i \, q_{i-1} , \quad i = 1, 2, \ldots, N \quad \text{where} \tag{4}$$

$c_i > 0$ and unique, and

$$c_i = \alpha_0/\{1 - \alpha_1 - \alpha_{N-i} \prod_{j=1}^{N-i} c_{i+j} - \sum_{K=1}^{N-i-1} \alpha_{K+1} \prod_{j=1}^{K} c_{i+j}\}, \tag{5}$$

$$c_N = \alpha_0/(1 - \alpha_0) ,$$

$$c_0 = 1/\{1 + \sum_{K=1}^{N} \prod_{i=1}^{N} c_i\}$$

<u>Proof</u>: The proof proceeds by induction on i, i = N, N-1, ..., 1 where we solve the steady state equations (1) recursively and then apply the normalizing condition.

For $i = N$, we solve for the last equation in $\underline{q}' \underline{p} = \underline{q}'$,

$$q_N \alpha_0 + q_{N-1} \alpha_0 = q_N$$

Defining $c_N = \alpha_0/(1 - \alpha_0)$ we get $q_N = c_N q_{N-1}$ and the theorem is true for $i = N$. Assuming $q_K = c_K q_{K-1}$ for $K = N, N-1, \ldots, i+1$, and solving for the $(i+1)$st equation in the system of equations $\underline{q}' \underline{p} = \underline{q}'$ yields

$$\underline{q}' \cdot \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \\ \alpha_0 \\ \alpha_1 \\ \cdot \\ \cdot \\ \alpha_{N-i} \\ \alpha_{N-i} \end{bmatrix} = \underline{q}'$$

or $q_{i-1} \alpha_0 + q_i \alpha_1 + \ldots + q_{N-1} \alpha_{N-i} + q_N \alpha_{N-i} = q_i$. Noting that

$q_{i+K} = c_{i+K} q_{i+K-1} = \{\prod_{j=1}^{K} c_{i+j}\} q_i$, and rearranging we get

$$q_{i-1} \alpha_0 = q_i \{1 - \alpha_1 - \sum_{K=1}^{N-i-1} \alpha_{K+1} \prod_{j=1}^{K} c_{i+j} - \alpha_{N-i} \prod_{j=1}^{N-i} c_{i+j}\}$$

which leads to the form given in (5) and therefore the induction is complete.

Now applying the normalizing condition $\sum_{i=0}^{N} q_i = 1$ and substituting

$$q_i = c_1 c_2 \ldots c_i q_0, \text{ we get}$$

$$q_0 = 1/\{1 + \sum_{K=1}^{N} \prod_{i=1}^{K} c_i\} \equiv c_0.$$

It is known that $\{q_i\}_0^N$ can be uniquely determined by solving equations (1), hence their respective ratios $c_K = q_K/q_{K-1}$, $K = 1, 2, \ldots, N$ are also unique.

Define $r_i = c_{N-i+1} = q_{N-i+1}/q_{N-i}$, $i \geq 2$

$$r_1 = c_N = q_N/q_{N-1} \ , \tag{6}$$

$$r_2 = c_{N-1} = q_{N-1}/q_{N-2} \ , \ \ldots$$

Then from theorem 1,

$$r_i = \alpha_0/[1 - \alpha_1 - \alpha_{i-1} \prod_{j=1}^{i-1} c_{N-i+j+1} - \sum_{K=1}^{i-2} \alpha_{K+1} \prod_{j=1}^{K} c_{N-i+j+1}];$$

but $c_{N-i+j+1} = r_{i-j}$, and therefore

$$r_i = \alpha_0/[1 - \alpha_1 - \alpha_{i-1} \prod_{j=1}^{i-1} r_{i-j} - \sum_{K=1}^{i-2} \alpha_{K+1} \prod_{j=1}^{K} r_{i-j}], \ i \geq 2$$

$$r_1 = \alpha_0/(1 - \alpha_0) \tag{7}$$

The sequence r_i can be generated recursively and independently of the system capacity N. That is, the sequence $\{r_i\}$ for $G/M/1/N_1$ is the same as that for $G/M/1/N_2$, $G/M/1/N_3$, ... Hence the $\{r_i\}$ is a generating sequence for the class G/M/1/N with the same interarrival and service time distributions.

To study the behaviour of $G/M/1/N_j$ for different system capacities N_j, $\{r_i\}$ provides a computational convenience. It suffices to generate $\{r_i\}$ once as in (7) and for every $N \geq 1$, $\{q_i\}_0^N$ is given by

$$q_i = r_{N-i+1}\, q_{i-1}\ ,\quad i = 1, 2, \ldots, N$$

$$q_0 = 1/[1 + \sum_{i=1}^{N} \prod_{K=1}^{i} r_{N-K+1}] \tag{8}$$

Table 1 gives r_i, i = 1, 2, ..., 10 for four different interarrival time distrbutions (two erlangian, an exponential, and a hyper exponenetial distribution). Note that from this table steady state distributions for systems with the given interarrival and service time distributions and $N \leq 10$ can be determined based on the following lemma.

Table 1

cov	.447* K=5	.577* K=3	1.000 K=1	2.134** P_1=.1
r_1	3.169	3.684	4	5.049
r_2	5.668	5.252	4	3.118
r_3	6.781	5.984	4	2.094
r_4	7.334	6.328	4	1.784
r_5	7.576	6.487	4	1.707
r_6	7.669	6.556	4	1.689
r_7	7.699	6.585	4	1.685
r_8	7.705	6.597	4	1.684
r_9	7.706	6.601	4	1.684
r_{10}	7.706	6.603	4	1.684

*For Erlangian family, $dA(x) = \dfrac{e^{-\lambda Kx}(\lambda K)^K x^{K-1}}{(K-1)!}$, $\text{cov} = \dfrac{1}{\sqrt{K}}$

**For Hyperexponential family, $d\,A(x) = 2p_1\,\lambda\, e^{-p_1\lambda x} + 2p_2\,\lambda\, e^{-p_2\lambda x}$,

$p_1 = 1 - p_2$, $\text{cov} = \sqrt{\dfrac{1}{2p_1p_2} - 1}$

Lemma 1: Let $\{q_i^{(1)}\}$ and $\{q_i^{(2)}\}$ be the steady state distributions of the two systems G/M/I/N_1 and G/M/1/N_2 where all factors other than N_1 and N_2 are the same for the two systems. Then we have

$$q^{(1)}_{N_1-i+1} / q^{(1)}_{N_1-i} = q^{(2)}_{N_2-i+1} / q_{N_2-i}\ , \ N_1 - i,\ N_2 - i \geq 0 \tag{9}$$

Proof: Both sides of (9) are equal to r_i, where r_i is independent of N_1 and N_2.

The existence of the generating sequence $\{r_i\}$ can also be observed from the results of Raju [15], where a monotone nondecreasing sequence $\{a_i\}$ is used to calculate the steady state distribution for the G/M/1/N system. The derivation of these results is based on a fundamental recursion used in inverting the system of equations (1).

Theorem 2: [Raju and Bhat [15]]

The steady state arrival epoch distribution of the G/M/1/N system is given by:

$$q_j = \frac{a_{N-1-j} - a_{N-2-j}}{a_{N-1}}\ , \ q_{N-1} = \frac{1 - \alpha_0}{a_{N-1}}\ , \ q_N = \frac{\alpha_0}{a_{N-1}}$$

where

$$a_0 = 1$$

$$a_1 = (1 - \alpha_1)/\alpha_0$$

$$a_{K+1} = \frac{1}{\alpha_0} \{a_K - \sum_{t=0}^{K} \alpha_{t+1}\, a_{K-t}\} \quad , \quad K \geq 1$$

According to Theorem 2 and by virtue of the uniqueness of $\{r_i\}$, r_i will take the alternative form

$$r_1 = \frac{\alpha_0}{1 - \alpha_0}, \quad r_2 = \frac{\alpha_0(1 - \alpha_0)}{1 - \alpha_0 - \alpha_1}$$

$$r_i = \frac{a_{i-2} - a_{i-3}}{a_{i-1} - a_{i-2}}, \quad i \geq 3 \tag{10}$$

Often, one would like to increment N to N + 1 or decrement N to N - 1 to study corresponding effect on both the steady state distribution and other measures of performance. Theorem 2 gives rise to some interesting properties as in the following lemma.

Lemma 2: The steady state distribution for G/M/1/N+1 system is given by

$$q_i^{(N+1)} = q_{i-1}^{(N)} \frac{a_{N-1}}{a_N}, \quad i \geq 1,$$

and (11)

$$q_0^{(N+1)} = 1 - \frac{a_{N-1}}{a_N}$$

Proof: Using Theorem 2,

$$q_i^{(N+1)} = \frac{a_{N-i} - a_{N-1-i}}{a_N} = \frac{a_{N-1-(i-1)} - a_{N-2-(i-1)}}{a_{N-1}} \frac{a_{N-1}}{a_N}$$

$$= q_{i-1}^{(N)} \frac{a_{N-1}}{a_N}$$

$$1 = q_0^{(N+1)} + \sum_{i=1}^{N+1} q_i^{(N+1)} = q_0^{(N+1)} + \frac{a_{N-1}}{a_N} \sum_{i=1}^{N+1} q_{i-1}^{(N)}$$

$$= q_0^{(N+1)} + \frac{a_{N-1}}{a_N} \sum_{i=0}^{N} q_i^{(N)} = q_0^{N+1} + \frac{a_{N-1}}{a_N};$$

hence the result.

3. BOUNDS FOR THE MEASURE Z_N

Consider the optimization model

$$
\begin{aligned}
P_N: \quad & \min_{\underline{q} \geq 0} Z_N = \sum_{i=0}^{N} \bar{a}_i \, q_i = \bar{a}' \, \underline{q} \\
\text{s.t.} \quad & \underline{q}' \, \underline{P} = \underline{q}' \\
& \underline{q}' \, \underline{1} = 1
\end{aligned}
\tag{12}
$$

For a given general interarrival time distribution, μ, and N the system of equations (12) has a linear objective function and linear constraints in $\underline{q}$. The feasible region is a single point $\underline{q}^* \in E^{N+1}$ which is the solution to the steady state equations. It is known from the theory of linear programming [4] that a relaxation of the feasible region yields a solution which is a lower bound to the objective function Z_N of (12). A specialization of this approach is used in Kotiah [12], to prove that the probability of zero delay for M/D/2 is larger than that for M/M/2 with the same traffic intensity and to obtain bounds on the average queue length for an M/M/s mixed queue. Here we develop bounds on the generalized measure Z_N using a particular relaxation for P_N.

<u>Definition</u>: Let the i^{th} relaxation PR_i of P_N be defined as

$$
PR_i: \quad \min_{\underline{q} \geq 0} \bar{Z}_N = \bar{a}' \, \underline{q}
$$

s.t.

$$
(q_0, q_1, \ldots, q_N)
\left[\begin{array}{c|cccccc}
\underline{0} & & & & \underline{0} & & \\
\hline
 & \alpha_0 & & & & & \\
 & \alpha_1 & \alpha_0 & & & & \\
\underline{0} & \cdot & & & & & \\
 & \cdot & & & & & \\
 & \alpha_{i-1} & \cdot & \cdot & \cdot & \alpha_1 & \alpha_0 \\
 & \alpha_{i-1} & \cdot & \cdot & \cdot & \alpha_1 & \alpha_0
\end{array}\right]
= (q_0, q_1, \ldots q_{N-i+1}, \ldots q_N)
\tag{13}
$$

$$\sum_{i=0}^{N} q_1 = 1 \quad , \quad q_i \geq 0$$

Thus the i^{th} relaxation reduces the constraint set to the last i constraints along with the normalizing condition.

Using the sequence $\{r_i\}$ and Theorem 1 we can write PR_i as

$$\begin{aligned} PR_i: \ \min_{\underline{q}\geq 0} \ \overline{Z}_N = \overline{a}' \underline{q} \\ \text{s.t.} \qquad r_1 q_{N-1} - q_N &= 0 \\ r_2 q_{N-2} - q_{N-1} &= 0 \\ &\vdots \\ r_i q_{N-i} - q_{N-i+1} &= 0 , \\ \sum_{i=0}^{N} q_i &= 1 \end{aligned} \tag{14}$$

The constraints in (14) are equality constraints and can be substituted in the objective function to give the concise form of PR_i.

$$\begin{aligned} PR_i: \ \min_{\underline{q}\geq 0} \ \overline{Z}_N &= \sum_{i=0}^{N-i-1} \overline{a}_i q_i + q_{N-i} \overline{h}_i \\ \text{s.t.} \quad \sum_{i=0}^{N-i-1} q_i + q_{N-i} h_i &= 1 \end{aligned} \tag{15}$$

where

$$\overline{h}_i = \overline{a}_{N-i} + \overline{a}_{N-i+1} r_i + \ldots + \overline{a}_N r_i r_{i-1} \ldots r_1 ,$$

$$h_i = 1 + r_i + r_i r_{i-1} + \ldots + r_i r_{i-1} \ldots r_1 ,$$

where $\{r_i\}$ are known for the class G/M/1/N, $N \geq 1$. Now any solution of (15) will provide a lower bound on Z_N.

Definition: Define the i^{th} lower bound LB_i of Z_N as the optimal solution to (15). Theorem 3 gives the explicit form of the lower bounds.

Theorem 3:

$$Z_N \geq LB_i = \min\{\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{N-i-1}, \bar{h}_i/h_i\}, \; N > i \geq 1$$

Proof: PR_i is a linear programming primal problem with one constraint. Defining DR_i as the dual problem of PR_i with w as the dual variable, we have:

$$\begin{aligned} DR_i: \quad & \max_w \; 1.\, w \\ \text{s.t.} \quad & w \leq \bar{a}_0 \\ & w \leq \bar{a}_1 \\ & \vdots \\ & w \leq \bar{a}_{N-i-1} \\ & h_i\, w \leq \bar{h}_i \end{aligned}$$

By simple inspection, the optimal feasible solution to DR_i is

$$w^* = \min\{\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{N-i-1}, \bar{h}_i/h_i\}, \; h_i > 0$$

Also since w^* is optimal feasible to DR_i, an optimal solution $\bar{Z}_i^*$ to the primal PR_i exists (see [4]) and

$$w^* = \bar{Z}_i^* \leq Z_N$$

Definition: Define the i^{th} upper bound UB_i of Z_N as the optimal solution to the following:

$$\max_{q \geq 0} \bar{Z}_N = \sum_{i=0}^{N-i-1} \bar{a}_i \, q_i + q_{N-i} \, \bar{h}_i$$

$$\text{s.t.} \quad \sum_{i=0}^{N-i-1} q_i + q_{N-i} \, h_i = 1 \tag{16}$$

Theorem 4 gives the form of the upper bounds.

<u>Theorem 4</u>:

$$Z_N \leq UB_i = \max \{\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_{N-i-1}, \bar{h}_i / h_i\}$$

Proof of this theorem is similar to that of Theorem 3.

Theorem 5 shows that the sequence LB_i, $i = 1, 2, \ldots, N$ is monotone increasing and converges to Z_N for $i = N$.

<u>Theorem 5</u>:

$$\bar{Z}^*_{i-1} \leq \bar{Z}^*_i \leq Z_N \, , \quad 2 \leq i \leq N-1$$

with

$$\bar{Z}^*_N = Z_N$$

<u>Proof</u>: By Theorem 3:

$$Z_N \geq \bar{Z}^*_i \qquad \text{for all } N-1 \geq i \geq 1$$

Let F_i be the set of all feasible solutions to PR_i, and let F be the set of all feasible solutions to P_N, then since PR_{i-1} is a relaxation to PR_i,

$$F \subseteq F_i \subseteq F_{i-1}$$

with

$$F = F_N,$$

hence

$$Z_N \geq \bar{Z}_i^* \geq \bar{Z}_{i-1}^*$$

and

$$Z_N = \bar{Z}_N^*$$

A similar result can be obtained for the convergence of the sequence of upper bounds UB_i to Z_N.

Thus Z_N is shown to be trapped in a sequence of intervals that can be made arbitrarily small by virtue of the convergence property,

$$\min_{0 \leq J \leq N-i} A_j \leq Z_N \leq \max_{0 \leq j \leq N-i} A_j, \; i = 1, 2, \ldots, N \qquad (17)$$

$$A_j = \begin{cases} \bar{a}_j & 0 \leq j \leq N-i-1 \\ \bar{h}_i/h_i & j = N-i \end{cases}$$

To illustrate the applicability of the bounds consider the following profit functions:

(1) The coefficients $\{\bar{a}_j\}$ are monotone increasing in j, the number of customers in the system, e.g.,

$$Z_1 = \sum_{j=0}^{N} j^x q_j \; , \; x > 0$$

(2) $\{\bar{a}_j\}$ is monotone decreasing sequence in j, the number of customers in the system, e.g.,

$$Z_2 = \sum_{j=0}^{N} (1 - jy)\, q_j \;, \; y > 0$$

(3) $\{\bar{a}_j\}$ is the sum of the above two sequences,

$$Z_3 = Z_1 + Z_2$$

Table 2 shows the bounds to Z_1, Z_2, Z_3 for $E_5/M/1/5$ system with x = .4, y = .2 for $\rho = \lambda\mu^{-1}$ = .8, 2, 4, 10. The generating sequence $\{r_i\}$ is given for each ρ. The results show that the bounds are tighter under heavy traffic conditions since they rely more on q_N, q_{N-1}, ... rather than q_0, q_1, ... They also show that, in general, we should expect either the upper or the lower bounds to be useful.

The form (17) suggests different approximations to Z_N. For example $Z_N \simeq 1/2(LB_i + UB_i)$ where the error will be known not to exceed $1/2(UB_i - LB_i)$, $1 \leq i \leq N-1$. However, an epproximation of a particular importance and interpretation is the $Z_N \simeq A_{N-i} = \bar{h}_i/h_i$ as shown in the following section.

Table 2

ρ	i	r_i	Z_1 (LB_i, UB_i)	Z_2 (LB_i, UB_i)	Z_3 (LB_i, UB_i)
.8	1	.487	(0.000, 1.792)	(.134, 1.000)	(1.000, 1.950)
	2	.639	(0.000, 1.668)	(.270, 1.000)	(1.000, 1.938)
	3	.675	(0.000, 1.517)	(.412, 1.000)	(1.000, 1.930)
	4	.681	(0.000, 1.317)	(.562, 1.000)	(1.000, 1.880)
	5	.682	(0.840, 0.840)	(.721, 0.721)	(1.561, 1.561)
2	1	1.638	(0.000, 1.839)	(.075, 1.000)	(1.000, 1.950)
	2	2.430	(0.000, 1.800)	(.119, 1.000)	(1.000, 1.920)
	3	2.769	(0.000, 1.778)	(.141, 1.000)	(1.000, 1.919)
	4	2.891	(0.000, 1.765)	(.152, 1.000)	(1.000, 1.918)
	5	2.929	(1.756, 1.756)	(.156, 0.156)	(1.913, 1.913)
4	1	3.619	(0.000, 1.865)	(.043, 1.000)	(1.000, 1.950)
	2	5.668	(0.000, 1.853)	(.056, 1.000)	(1.000, 1.920)
	3	6.781	(0.000, 1.851)	(.059, 1.000)	(1.000, 1.910)
	4	7.333	(0.000, 1.850)	(.060, 1.000)	(1.000, 1.910)
	5	7.576	(1.850, 1.850)	(.060, 0.060)	(1.910, 1.910)
10	1	9.607	(0.000, 1.884)	(.018, 1.000)	(1.000, 1.950)
	2	15.603	(0.000, 1.882)	(.021, 1.000)	(1.000, 1.920)
	3	19.421	(0.000, 1.882)	(.021, 1.000)	(1.000, 1.904)
	4	21.810	(0.000, 1.882)	(.021, 1.000)	(1.000, 1.904)
	4	23.233	(1.882, 1.882)	(.021, 0.021)	(1.904, 1.904)

4. APPROXIMATIONS TO THE GENERALIZED MEASURE Z_N

Here we develop a sequence of approximations for Z_N based on A_{N-i},

<u>Definition</u>: The i^{th} approximation $\hat{Z}_i$ $(i=1,2, \ldots, N)$ to Z_N is given by the solution to the system

$$\min_{\hat{q}\geq 0} \sum_{N-i}^{N} \bar{a}_j \hat{q}_j$$

s.t.

$$\left.\begin{aligned} r_1 \hat{q}_{N-1} - \hat{q}_N &= 0 \\ r_2 \hat{q}_{N-2} - \hat{q}_{N-1} &= 0 \\ &\vdots \\ r_i \hat{q}_{N-i} - \hat{q}_{N-i+1} &= 0 \end{aligned}\right\} \quad (18)$$

$$\hat{q}_{N-i} + \hat{q}_{N-i+1} + \cdots + \hat{q}_N = 1 \quad (19)$$

where $\hat{q}_j$ are the approximated steady state probabilities, $N \geq j \geq N - i$.

<u>Lemma 3</u>: $\hat{Z}_i = \bar{h}_i/h_i$ and the approximated steady state probabilities $\{\hat{q}_j\}$ are given by

$$\hat{q}_j = \begin{cases} q_j^{(N)} \cdot d & N - i \leq j \leq N \\ 0 & \text{otherwise.} \end{cases}$$

where $d = a_{N-1}/a_{i-1}$, and $\{q_j^{(N)}\}$ is the exact steady state solution to the corresponding G/M/1/N system.

Proof: First observe that equations (18) and (19) are the steady state equations for the corresponding G/M/1/i system expressed in terms of ratios $\{r_i\}$. Let $\{q_K^{(i)}\}_{K=0}^{i}$, be the solution to G/M/1/i system, then

$$\begin{aligned} r_1\, q_{i-1}^{(i)} - q_i^{(i)} &= 0 \\ r_2\, q_{i-2}^{(i)} - q_{i-1}^{(i)} &= 0 \\ &\vdots \\ r_i\, q_0^{(i)} - q_1^{(i)} &= 0 \\ q_0^{(i)} + q_1^{(i)} + \ldots + q_i^{(i)} &= 1 \end{aligned}$$

hence

$$\hat{q}_j = q_{j-(N-i)}^{(i)}$$

will satisfy (18) and (19).

Also from lemma 2:

$$q_j^{(N)} = q_{j-1}^{(N-1)} \frac{a_{N-2}}{a_{N-1}} = q_{j-2}^{(N-2)} \frac{a_{N-3}}{a_{N-1}} = \cdots = q_{j-K}^{(N-K)} \frac{a_{N-K-1}}{a_{N-1}}$$

letting $K = N - i$,

$$q_j^{(N)} = q_{j-(N-i)}^{(i)} \cdot \frac{a_{i-1}}{a_{N-1}} \quad \text{or}$$

$$\hat{q}_j = q_{j-(N-i)}^{(i)} = q_j^{(N)}\, d\ , \text{ where } d = \frac{a_{N-1}}{a_{i-1}}\ .$$

Also system of equation (18) implies

$$\hat{q}_{N-i} = r_i^{-1}\, \hat{q}_{N-i+1} = [r_i\, r_{i-1}]^{-1}\, \hat{q}_{N-i+2} = \cdots = [r_i\, r_{i-1} \ldots r_i]^{-1}\, \hat{q}_n$$

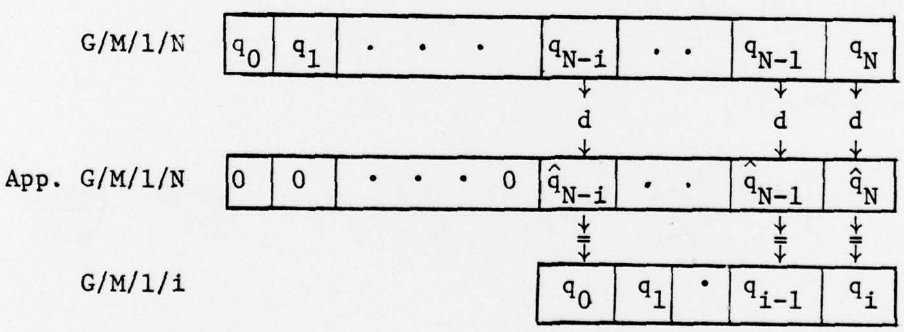
Substituting into (19) gives

$$\hat{q}_{N-1} = 1/h_i$$

and back substitution into the objective function gives

$$\hat{Z}_i = \bar{h}_i/h_i$$

Thus the i^{th} approximation can be viewed in different ways. The approximation uses a G/M/1/i system to evaluate the last $i + 1$ probabilities and sets $\hat{q}_j = 0$, $j < N - i$. Also the approximation overestimates the last $i + 1$ probabilities by multiplying them by $d = \frac{a_{N-1}}{a_{i-1}} \geq 1$ (Note the monotonicity of $\{a_j\}$). The following figure shows the relationships among the approximation, the G/M/1/N (exact) system, and the G/M/1/i system.



Thus the quality of the i^{th} approximation (to be measured quantitatively in the following section) depends on the factor d and the traffic intensity $\rho = \lambda/\mu$. The closer the d to 1 the better is the approximation. A heavy traffic situation justifies setting the first few probabilities to zero. It has been observed in experimentation that d is also related to the traffic intensity such that heavier traffic provides values of d which are

closer to 1. The following table shows d for different levels of approximations $i = 1, 2, 3, .., N - 1$ for Erlangian systems ($0 < cov \leq 1$), and Hyperexponential systems ($1 < cov < \infty$) under various traffic loads.

λ	u	cov	N	d									
				i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	i=9	i=10
2	1	.447	10	1.241	1.074	1.008	1.008	1.002	1.000	→			
4	1	.447	10	1.044	1.006	1.000	→						
10	1	.447	10	1.006	1.000	→							
2	1	1.000	10	1.332	1.142	1.006	1.031	1.015	1.007	1.003	1.001	1.000	1.000
4	1	1.000	10	1.066	1.015	1.003	1.001	1.000	→				
10	1	1.000	10	1.010	1.000	→							
2	1	2.134	10	1.416	1.255	1.173	1.121	1.084	1.057	1.036	1.021	1.009	1.000
4	1	2.134	10	1.112	1.056	1.031	1.017	1.010	1.005	1.002	1.001	1.000	1.000
10	1	2.134	10	1.021	1.000	→							

Using the bounds (17) it can be easily shown that the approximation also bounds the measure Z_N such that

$$\hat{Z}_i \geq Z_N \qquad \text{if} \qquad \hat{Z}_i \geq \max_{0 \leq j \leq N-i-1} (\bar{a}_j),$$

and

$$\hat{Z}_i \leq Z_N \qquad \text{if} \qquad \hat{Z}_i \leq \min_{0 \leq j \leq N-i-1} (\bar{a}_j).$$

The following lemma shows that if $\{\bar{a}_j\}$ is a monotone increasing sequence the approximation will bound Z_N from above. An important special case is whenever $\bar{a}_j = j^n$, $n = 1, 2, 3, \ldots$ which gives the n^{th} raw moment of the distribution of customers in the system.

Lemma 4: Let $\{\bar{a}_j\}$ be a monotone increasing sequence, then

$$Z_N \leq \hat{Z}_i \quad , \quad i \leq N - 1$$

Proof:

$$Z_N = \sum_{j=0}^{N} \bar{a}_j q_j = \sum_{j=0}^{N-i-1} \bar{a}_j q_j + \sum_{j=N-i}^{N} \bar{a}_j q_j$$

$$\leq \bar{a}_{N-i-1} \left(1 - \sum_{j=N-i}^{N} q_j\right) + \sum_{j=N-i}^{N} \bar{a}_j q_j$$

$$= \bar{a}_{N-i-1} + \sum_{j=N-i}^{N} (\bar{a}_j - \bar{a}_{N-i-1}) q_j$$

$$\leq \bar{a}_{N-i-1} + \sum_{j=N-i}^{N} (\bar{a}_j - \bar{a}_{N-i-1}) \hat{q}_j, \text{ by lemma 3}$$

$$= \bar{a}_{N-i-1} - \bar{a}_{N-i-1} \sum_{j=N-i}^{N} \hat{q}_j + \hat{Z}_i$$

$$= \hat{Z}_i \quad , \quad \text{since} \sum_{j=N-i}^{N} \hat{q}_j = 1.$$

5. COMPUTATIONAL EXPERIENCE

The approximation procedure as presented in the previous section has been coded and used to obtain computational experience for $E_K/M/1/N$ (K-Erlangian) systems and $H_p/M/1/N$ (Hyperexponential) systems. These two families of queues are commonly used in the literature since they cover the full range of the coefficient of variation (cov) and they provide a generating formula for the elements $\alpha_j, j = 0, 1, \ldots, N$. This specialization does not simplify the approximation procedure but does simplify the coding effort.

For different combinations of N, λ, μ, and cov, a set of 100 problems were solved for the i^{th} approximation $i = 1, 2, .., N - 1$. The cost coefficients $\{\bar{a}_j\}$ were randomly generated as uniform (0, 1) variables for each problem. Thus the resulting generalized measures Z_N, $\hat{Z}_i$ are on a (0, 1) scale. The choice of the different parameters was designed to reflect the effect of traffic intensity, coefficient of variation, and system capacity (N) on the quality of the approximation.

Two measures were used to evaluate the i^{th} approximation, the absolute deviation $|Z_N - \hat{Z}_i|$, and the percent deviation $|Z_N - \hat{Z}_i|/Z_N$, $i = 1, 2, .., N - 1$. For each problem set and each i, the mean and variance of the above two measures are given as

DEV (I)	mean abs. deviation
SD (I)	variance abs. deviation
PDEV (I)	mean percent deviation
SP (I)	variance of percent deviation

Scaling the cost coefficients should not affect the percent deviation while it should scale the absolute deviation with the same factor. Tables 3, 4, and 5 present our computational experience.

Clearly the approximation works best under heavy traffic loads. For $\rho = 5$, the 2^{nd} approximation ($i = 2$) gives rise to a max deviation of .0082 and a max percent deviation of 2.5%; for $\rho = 10$, max deviation drops to .0019 and max percent deviation drops to .6% over the range of cov and N considered. This fact combined with the simple formula for this approximation

$$\hat{Z}_2 = \frac{\bar{a}_{N-2} + r_2 \bar{a}_{N-1} + r_2 r_1 \bar{a}_N}{1 + r_2 + r_2 r_1}$$

makes it an efficient approximation for heavy traffic. For medium traffic, similar results can be obtained for $i = 3$, $i = 4$. Over the range of cov, it seems that the approximation performs better for smaller values of coefficient of variation. A comparison of table 6 with the appropriate tables 3, 4, and 5, shows that the approximation yields better results for smaller values of N.

As expected from the derivation the approximation can not handle low traffic conditions. Clearly an approximation that uses $q_0, q_1, q_2, \ldots$ should yield good results in which case the i^{th} approximation will be a function of $r_N, r_{N-1}, \ldots, r_{N-i}$ rather than $r_1, r_2, \ldots, r_i$.

Finally we believe that the simplicity of the formula and the derivation of the bounds, the approximations, and the generating ratios for the G/M/1/N system have great potential in the optimization of this class of queues.

Table 3: Hp/M/1/10 System

cov*	λ	μ	I	DEV(I)	SDOIP	PDEV(I)	SP(I)
2.134	1.	1.	2	.1006	.0055	.2043	.0245
			3	.0790	.0034	.1582	.0131
			4	.0638	.0020	.1299	.0089
			5	.0516	.0014	.1055	.0061
			6	.0432	.0011	.0885	.0050
			7	.0305	.0005	.0618	.0022
			8	.0240	.0004	.0493	.0016
			9	.0175	.0001	.0352	.0006
			10	.0000	.0000	.0000	.0000
2.134	3.	1.	2	.0209	.0002	.0504	.0024
			3	.0117	.0001	.0274	.0008
			4	.0073	.0000	.0180	.0004
			5	.0045	.0000	.0106	.0002
			6	.0029	.0000	.0070	.0001
			7	.0018	.0000	.0044	.0000
			8	.0010	.0000	.0025	.0000
			9	.0005	.0000	.0013	.0000
			10	.0000	.0000	.0000	.0000
2.134	5.	1.	2	.0082	.0000	.0246	.0010
			3	.0040	.0000	.0123	.0003
			4	.0022	.0000	.0067	.0001
			5	.0010	.0000	.0030	.0000
			6	.0005	.0000	.0016	.0000
			7	.0002	.0000	.0007	.0000
			8	.0001	.0000	.0002	.0000
			9	.0001	.0000	.0002	.0000
			10	.0000	.0000	.0000	.0000
2.134	10.	1.	2	.0017	.0000	.0057	.0001
			3	.0006	.0000	.0022	.0000
			4	.0002	.0000	.0008	.0000
			5	.0001	.0000	.0003	.0000
			6	.0000	.0000	.0001	.0000
			7	.0000	.0000	.0000	.0000
			8	.0000	.0000	.0000	.0000
			9	.0000	.0000	.0000	.0000
			10	.0000	.0000	.0000	.0000

*cov = coefficient of variation.

Table 4: M/M/1/10 System

cov	λ	μ	I	DEV(I)	SD(I)	PDEV(I)	SP(I)
1.	1.	1.	2	.1190	.0062	.2410	.0271
			3	.0898	.0039	.1820	.0180
			4	.0783	.0033	.1595	.0154
			5	.0651	.0022	.1333	.0109
			6	.0489	.0014	.1016	.0071
			7	.0443	.0009	.0917	.0042
			8	.0355	.0006	.0730	.0028
			9	.0231	.0002	.0476	.0011
			10	.0000	.0000	.0000	.0000
1.	3.	1.	2	.0088	.0000	.0230	.0008
			3	.0028	.0000	.0072	.0002
			4	.0010	.0000	.0027	.0002
			5	.0003	.0000	.0009	.0001
			6	.0001	.0000	.0003	.0001
			7	.0000	.0000	.0001	.0000
			8	.0000	.0000	.0000	.0000
			9	.0000	.0000	.0000	.0000
			10	.0000	.0000	.0000	.0000
1.	5.	1.	2	.0020	.0000	.0061	.0001
			3	.0005	.0000	.0015	.0000
			4	.0001	.0000	.0003	.0000
			5	.0000	.0000	.0001	.0000
			6	.0000	.0000	.0000	.0000
			7	.0000	.0000	.0000	.0000
			8	.0000	.0000	.0000	.0000
			9	.0000	.0000	.0000	.0000
			10	.0000	.0000	.0000	.0000
1.	10.	1.	2	.0003	.0000	.0011	.0000
			3	.0000	.0000	.0001	.0000
			4	.0000	.0000	.0000	.0000
			5	.0000	.0000	.0000	.0000
			6	.0000	.0000	.0000	.0000
			7	.0000	.0000	.0000	.0000
			8	.0000	.0000	.0000	.0000
			9	.0000	.0000	.0000	.0000
			10	.0000	.0000	.0000	.0000

Table 5: E_2/M/1/10 System

cov	λ	μ	I	DEV(I)	SD(I)	PDEV(I)	SP(I)
.707	1.	1.	2	.1232	.0071	.2512	.0318
			3	.0966	.0054	.1981	.0239
			4	.0776	.0036	.1583	.0170
			5	.0602	.0023	.1229	.0102
			6	.0510	.0015	.1041	.0070
			7	.0439	.0009	.0894	.0045
			8	.0342	.0007	.0708	.0039
			9	.0228	.0002	.0468	.0009
			10	.0000	.0000	.0000	.0000
.707	3.	1.	2	.0054	.0000	.0125	.0004
			3	.0015	.0000	.0034	.0003
			4	.0004	.0000	.0009	.0002
			5	.0001	.0000	.0002	.0001
			6	.0000	.0000	.0001	.0001
			7	.0000	.0000	.0000	.0000
			8	.0000	.0000	.0000	.0000
			9	.0000	.0000	.0000	.0000
			10	.0000	.0000	.0000	.0000
.707	5.	1.	2	.0014	.0000	.0039	.0000
			3	.0002	.0000	.0005	.0000
			4	.0000	.0000	.0001	.0000
			5	.0000	.0000	.0000	.0000
			6	.0000	.0000	.0000	.0000
			7	.0000	.0000	.0000	.0000
			8	.0000	.0000	.0000	.0000
			9	.0000	.0000	.0000	.0000
			10	.0000	.0000	.0000	.0000
.707	10.	1.	2	.0002	.0000	.0007	.0000
			3	.0000	.0000	.0000	.0000
			4	.0000	.0000	.0000	.0000
			5	.0000	.0000	.0000	.0000
			6	.0000	.0000	.0000	.0000
			7	.0000	.0000	.0000	.0000
			8	.0000	.0000	.0000	.0000
			9	.0000	.0000	.0000	.0000
			10	.0000	.0000	.0000	.0000

Table 6: G/M/1/5 Systems

cov	λ	μ	I	DEV(I)	SD(I)	PDEV(I)	SP(I)
.707	3.	1.	2	.0060	.0000	.0177	.0007
			3	.0013	.0000	.0037	.0000
			4	.0003	.0000	.0009	.0000
			5	.0000	.0000	.0000	.0000
.707	5.	1.	2	.0013	.0000	.0036	.0000
			3	.0002	.0000	.0006	.0000
			4	.0000	.0000	.0001	.0000
			5	.0000	.0000	.0000	.0000
.707	10.	1.	2	.0002	.0000	.0005	.0000
			3	.0000	.0000	.0000	.0000
			4	.0000	.0000	.0000	.0000
			5	.0000	.0000	.0000	.0000
1.000	3.	1.	2	.0083	.0000	.0208	.0009
			3	.0025	.0000	.0066	.0002
			4	.0009	.0000	.0023	.0001
			5	.0000	.0000	.0000	.0000
1.000	5.	1.	2	.0022	.0000	.0062	.0001
			3	.0004	.0000	.0012	.0000
			4	.0001	.0000	.0002	.0000
			5	.0000	.0000	.0000	.0000
1.000	10.	1.	2	.0003	.0000	.0009	.0000
			3	.0000	.0000	.0001	.0000
			4	.0000	.0000	.0000	.0000
			5	.0000	.0000	.0000	.0000
2.134	3.	1.	2	.0186	.0001	.0507	.0025
			3	.0096	.0001	.0269	.0011
			4	.0050	.0000	.0141	.0002
			5	.0000	.0000	.0000	.0000
2.134	5.	1.	2	.0081	.0000	.0238	.0009
			3	.0037	.0000	.0111	.0002
			4	.0013	.0000	.0041	.0000
			5	.0000	.0000	.0000	.0000
2.134	10.	1.	2	.0019	.0000	.0069	.0001
			3	.0006	.0000	.0022	.0000
			4	.0002	.0000	.0005	.0000
			5	.0000	.0000	.0000	.0000

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A generalized measure of performance is defined as a weighted combination of the ergodic queue length distribution where the weights are general functions of the system parameters. The paper presents a sequence of upper and lower bounds for this measure of performance in the G/M/1/N queue with FIFO discipline. The bounds are used to derive a sequence of approximations with bounded errors. The upper and lower bounds are shown to converge to their corresponding exact values. The technique used is based on the

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imbedded Markov chain analysis and considers only subsets of the steady state equations to derive the bounds. Initial computational experience is encouraging and has indicated that the approximations are viable for heavy and medium traffic conditions.